HOSSAM GHANEM

(34) 11.1 11.2 Calculus with parametric curves

If $\tan\theta = \frac{a}{b}$ and $\tan\alpha = \left|\frac{a}{b}\right|$

$0 \le \theta \le 2\pi$		
	Q	θ
$\frac{a}{b} > 0$	1	$\theta = \alpha$
$\frac{a}{b} < 0$	2	$\theta = \pi - \alpha$
$\frac{a}{b} > 0$	3	$\theta = \pi + \alpha$
$\frac{a}{b} < 0$	4	$\theta = 2\pi - \alpha$

$-\pi \le \theta \le \pi$		
	Q	θ
$\frac{a}{b} > 0$	1	$\theta = \alpha$
$\frac{a}{b} < 0$	2	$\theta = \pi - \alpha$
$\frac{a}{b} > 0$	3	$\theta = -\pi + \alpha$
$\frac{a}{b} > 0$	4	$\theta = -\alpha$

Formula	Example
If $x = f(t)$, $y = g(t)$	If $x = \sin t$, $y = t^2$
Then $\frac{dx}{dt} = f^{\backslash}(t)$, $\frac{dy}{dt} = g^{\backslash}(t)$	Then $\frac{dx}{dt} = \cos t$, $\frac{dy}{dt} = 2t$
OR $dx = f^{\backslash}(t)\,dt$, $dy = g^{\backslash}(t)\,dt$	OR $dx = \cos t\ dt$, $dy = 2t\ dt$

Formula		Example	
$\frac{dy}{dx}$	$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$		$\frac{dy}{dx} = \frac{2t}{\cos t}$
Slope of the tangent at $t = a$	$m = \left.\frac{dy}{dx}\right\|_{t=a}$	Slope of tangent at $t = \pi$	$m = \left.\frac{dy}{dx}\right\|_{t=\pi} = \frac{2\pi}{-1} = -2\pi$
The tangent is horizontal at	$\frac{dy}{dt} = 0$ *And* $\frac{dx}{dt} \ne 0$	The tangent is horizontal	$\frac{dy}{dt} = 2t$ $2t = 0$ $t = 0$
The tangent is vertical at	$\frac{dx}{dt} = 0$	The tangent is vertical	$\frac{dx}{dt} = \cos t$ $\cos t = 0$ $t = \frac{\pi}{2}, \frac{3\pi}{2}$

$\frac{d^2y}{dx^2}$	$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt}$		$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt}$ $= \frac{d}{dt}\left(\frac{2t}{\cos t}\right) \div \cos t$ $= \left(\frac{2\cos t + 2t\sin x}{\cos^2 t}\right) \cdot \frac{1}{\cos t}$
The curve is Concave upward If	$\frac{d^2y}{dx^2} > 0$	Minimum Local Extrema	$\frac{dy}{dx} = 0$ And $\frac{d^2y}{dx^2} > 0$
The curve is Concave downward If	$\frac{d^2y}{dx^2} < 0$	Maximum Local Extrema	$\frac{dy}{dx} = 0$ And $\frac{d^2y}{dx^2} < 0$
Inflection Point if	$\frac{d^2y}{dx^2} = 0$	Not's $\frac{dy}{dx} = 0$ if	$\frac{dy}{dt} = 0$ *And* $\frac{dx}{dt} \neq 0$

Example

$x = f(t) = t - \sin t$, $y = g(t) = 1 - e^t$

$\therefore dx = (1 - \cos t)dt$, $dy = e^t\, dt$

$\therefore \frac{dx}{dt} = (1 - \cos t)$, $\frac{dy}{dt} = e^t$

Area under the arc

$$A = \int_a^b y\,dx \rightarrow A = \int_\propto^\beta g(t)\, f^\backslash(t)dt$$

Example

$$A = \int_\propto^\beta (1 - e^t)\,(1 - \cos t)dt$$

Arc Length

$$L = \int_\propto^\beta \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

Example

$$L = \int_\propto^\beta \sqrt{(e^t)^2 + (1 - \cos t)^2}dt$$

Surface Area

$$S = 2\pi \int_\propto^\beta y \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

Example

$$S = 2\pi \int_\propto^\beta (1 - e^t) \sqrt{(e^t)^2 + (1 - \cos t)^2}dt$$

the Centroid

$$\overline{x} = \frac{1}{A}\int_\propto^\beta xy \cdot \frac{dx}{dt}dt$$

$$\overline{y} = \frac{1}{2A}\int_\propto^\beta y^2 \cdot \frac{dx}{dt}dt$$

Example 1

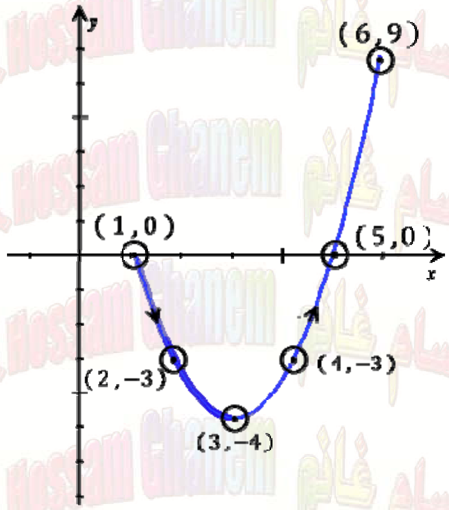
41 14 January 2012

Sketch the parametric curve $x = 1 + t$ and $y = t^2 - 4t$ for $0 \le t \le 5$. Indicate with an arrow the direction in which the curve is traced as t increases . (4 pts)

Solution

$x = 1 + t$, $y = t^2 - 4t$

t	x	y
0	1	0
1	2	−3
2	3	−4
3	4	−3
4	5	0
5	6	9



$t = x - 1$, $y = t^2 - 4t$

$y = (x-1)^2 - 4(x-1)$

$y = x^2 - 2x + 1 - 4x + 4$

Example 2

12 January 1998

If the curve C is given parametrically as

$x(t) = \sin^{-1}(e^{-t})\,,\ y(t) = \sec^{-1}(e^{t})$ $0 \le t \le \ln\sqrt{10}$ then find the length of C

$y = x^2 - 6x + 5$

Solution

$x = \sin^{-1} e^{-t}$

$$\frac{dx}{dt} = \frac{-e^{-t}}{\sqrt{1-e^{-2t}}} == \frac{1}{\sqrt{e^{2t}-1}}$$

$y = \sec^{-1} e^{t}$

$$\frac{dy}{dt} = \frac{e^{t}}{e^{t}\sqrt{e^{2t}-1}} = \frac{1}{\sqrt{e^{2t}-1}}$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{-e^{-2t}}{1-e^{-2t}} = \frac{1}{e^{2t}-1}$$

$$\left(\frac{dy}{dt}\right)^2 = \frac{1}{e^{2t}-1}$$

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = \frac{1}{e^{2t}-1} + \frac{1}{e^{2t}-1} = \frac{2}{e^{2t}-1}$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt = \int_0^{\ln 10} \sqrt{\frac{2}{e^{2t}-1}}\, dt = \sqrt{2}\int_0^{\ln 10} \frac{1}{\sqrt{e^{2t}-1}}\, dt$$

$$= \sqrt{2}\left[\sec^{-1} e^{t}\right]_0^{\ln 10} = \sqrt{2}\,[\sec^{-1} 10 - \sec^{-1} 1] = \sqrt{2}\,\sec^{-1} 10$$

Example 3

29 January 2007 A

Let the curve Γ: $x = e^t \cos t$, $y = e^t \sin t$, $. t \in [\,0\,, 2\pi\,]$

a. Find the equation of the tangent line at the point corresponding to $t = \dfrac{\pi}{2}$.

b. Find the points at which the tangent lines to Γ are vertical and those where the tangent lines are horizontal.

c. Find the length of Γ.

Solution

$x = e^t \cos t$ $\qquad$ $y = e^t \sin t$

$\dfrac{dx}{dt} = e^t \cos t - e^t \sin t$ $\qquad$ $\dfrac{dy}{dt} = e^t \sin t + e^t \cos t$

equation of the tangent

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{e^t \sin t + e^t \cos t}{e^t \cos t - e^t \sin t}$$

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{e^{\frac{\pi}{2}} \sin\frac{\pi}{2} + e^{\frac{\pi}{2}} \cos\frac{\pi}{2}}{e^{\frac{\pi}{2}} \cos\frac{\pi}{2} - e^t \sin\frac{\pi}{2}} = \frac{e^{\frac{\pi}{2}} + 0}{0 - e^{\frac{\pi}{2}}} = -1$$

$\sin\dfrac{\pi}{2} = 1$

$\cos\dfrac{\pi}{2} = 0$

$x|_{t=\frac{\pi}{2}} = e^{\frac{\pi}{2}} \cos\dfrac{\pi}{2} = 0$

$y|_{t=\frac{\pi}{2}} = e^{\frac{\pi}{2}} \sin\dfrac{\pi}{2} = e^{\frac{\pi}{2}}$

$$m = -1 \quad , \quad p\left(0\,, e^{\frac{\pi}{2}}\right)$$

$$y - y_1 = m(x - x_1)$$

$$y - e^{\frac{\pi}{2}} = -x$$

$$y + x - e^{\frac{\pi}{2}} = 0$$

Vertical tangent at $\dfrac{dx}{dt} = 0$

$$e^t \cos t - e^t \sin t = 0$$

$$\cos t - \sin t = 0$$

$$\sin t = \cos t$$

$$\tan t = 1$$

$$\alpha = \frac{\pi}{4}$$

1 $\quad t = \alpha = \dfrac{\pi}{4}$ $\qquad$ 3 $\quad t = \pi + \alpha = \pi + \dfrac{\pi}{4} = \dfrac{5\pi}{4}$

$t = \dfrac{\pi}{4} \rightarrow$ $\quad x|_{t=\frac{\pi}{4}} = e^{\frac{\pi}{4}} \cos\dfrac{\pi}{4} = e^{\frac{\pi}{4}} \cdot \dfrac{1}{\sqrt{2}} = \dfrac{e^{\frac{\pi}{4}}}{\sqrt{2}}$ $\quad y|_{t=\frac{\pi}{4}} = e^{\frac{\pi}{4}} \sin\dfrac{\pi}{4} = e^{\frac{\pi}{4}} \cdot \dfrac{1}{\sqrt{2}} = \dfrac{e^{\frac{\pi}{4}}}{\sqrt{2}}$

$t = \dfrac{5\pi}{4} \rightarrow$ $\quad x|_{t=\frac{5\pi}{4}} = e^{\frac{5\pi}{4}} \cos\dfrac{5\pi}{4} = e^{\frac{5\pi}{4}} \cdot \dfrac{-1}{\sqrt{2}} = \dfrac{-e^{\frac{5\pi}{4}}}{\sqrt{2}}$ $\quad y|_{t=\frac{5\pi}{4}} = e^{\frac{5\pi}{4}} \sin\dfrac{5\pi}{4} = e^{\frac{5\pi}{4}} \cdot \dfrac{-1}{\sqrt{2}} = \dfrac{-e^{\frac{5\pi}{4}}}{\sqrt{2}}$

$$\therefore \; V.T \;\; at \;\; \left(\frac{e^{\frac{\pi}{4}}}{\sqrt{2}}, \frac{e^{\frac{\pi}{4}}}{\sqrt{2}}\right) \;\; \& \;\; \left(\frac{-e^{\frac{5\pi}{4}}}{\sqrt{2}}, \frac{-e^{\frac{5\pi}{4}}}{\sqrt{2}}\right)$$

Horizontal tangent at $\frac{dy}{dt} = 0$ & $\frac{dx}{dt} \neq 0$

$$e^t \sin t + e^t \cos t = 0$$
$$\sin t + \cos t = 0$$
$$\sin t = -\cos t$$
$$\tan t = -1$$
$$\alpha = \frac{\pi}{4}$$

$\boxed{2}$

$$t = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$\boxed{4}$

$$t = 2\pi - \alpha = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

$t = \frac{3\pi}{4}$ $\rightarrow$ $x|_{t=\frac{3\pi}{4}} = e^{\frac{3\pi}{4}} \cos\frac{3\pi}{4} = e^{\frac{3\pi}{4}} \cdot \frac{-1}{\sqrt{2}} = \frac{-e^{\frac{3\pi}{4}}}{\sqrt{2}}$

$$y|_{t=\frac{3\pi}{4}} = e^{\frac{3\pi}{4}} \sin\frac{3\pi}{4} = e^{\frac{3\pi}{4}} \cdot \frac{1}{\sqrt{2}} = \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}}$$

$t = \frac{5\pi}{4}$ $\rightarrow$ $x|_{t=\frac{7\pi}{4}} = e^{\frac{7\pi}{4}} \cos\frac{7\pi}{4} = e^{\frac{7\pi}{4}} \cdot \frac{1}{\sqrt{2}} = \frac{e^{\frac{7\pi}{4}}}{\sqrt{2}}$

$$y|_{t=\frac{7\pi}{4}} = e^{\frac{7\pi}{4}} \sin\frac{7\pi}{4} = e^{\frac{7\pi}{4}} \cdot \frac{-1}{\sqrt{2}} = \frac{-e^{\frac{7\pi}{4}}}{\sqrt{2}}$$

$$\therefore \quad H.T \quad at \quad \left(\frac{-e^{\frac{3\pi}{4}}}{\sqrt{2}}, \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}}\right) \quad \& \quad \left(\frac{e^{\frac{7\pi}{4}}}{\sqrt{2}}, \frac{-e^{\frac{7\pi}{4}}}{\sqrt{2}}\right)$$

length of Γ

$$\left(\frac{dx}{dt}\right)^2 = e^{2t}\cos^2 t + e^{2t}\sin^2 t - 2e^{2t}\sin t\cos t$$

$$\left(\frac{dy}{dt}\right)^2 = e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2e^{2t}\sin t\cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t}(\cos^2 t + \sin^2 t) + e^{2t}(\sin^2 t + \cos^2 t) = 2e^{2t}$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \int_0^{2\pi} \sqrt{2e^{2t}}\, dt = \int_0^{2\pi} \sqrt{2}e^t\, dt = \sqrt{2}\Big[\, e^t \,\Big]_0^{2\pi} = \sqrt{2}[e^{2\pi} - e^0] = \sqrt{2}[e^{2\pi} - 1]$$

Example 4

28 July 2006 A

Consider the curve C given by the parametric equations

$$x = e^{\frac{t}{2}} \cos t \qquad y = e^{\frac{t}{2}} \sin t, \qquad 0 \le t \le 2\pi$$

(a) Show that (2 points)

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{5}{4} e^t$$

(b) The curve C is revolved about the x-axis. Find area of the resulting surface.

Solution

$$x = e^{\frac{t}{2}} \cos t \qquad\qquad y = e^{\frac{t}{2}} \sin t$$

$$\frac{dx}{dt} = \frac{1}{2} e^{\frac{t}{2}} \cos t - e^{\frac{t}{2}} \sin t \qquad\qquad \frac{dy}{dt} = \frac{1}{2} e^{\frac{t}{2}} \sin t + e^{\frac{t}{2}} \cos t$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{1}{4} e^t \cos^2 t + e^t \sin^2 t - 2e^t \sin t \cos t$$

$$\left(\frac{dy}{dt}\right)^2 = \frac{1}{4} e^t \sin^2 t + e^t \cos^2 t + 2e^t \sin t \cos t$$

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = \frac{1}{4} e^t (\cos^2 t + \sin^2 t) + e^t (\sin^2 t + \cos^2 t) = \frac{1}{4} e^t + e^t = \frac{5}{4} e^t$$

$$S = 2\pi \int_a^b y \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt = 2\pi \int_0^{2\pi} e^{\frac{t}{2}} \sin t \sqrt{\frac{5}{4} e^t}\, dt = 2\pi \int_0^{2\pi} e^{\frac{t}{2}} \sin t \, \frac{\sqrt{5}}{2} e^{\frac{t}{2}}\, dt$$

$$S = 2\pi \cdot \frac{\sqrt{5}}{2} \int_0^{2\pi} e^t \sin t\, dt = \sqrt{5}\,\pi \int_0^{2\pi} e^t \sin t\, dt = 2\sqrt{5}\,\pi \int_0^{\pi} e^t \sin t\, dt$$

$$I = \int e^t \sin t\, dt$$

$$u = e^t \qquad\qquad dv = \sin t\, dt$$

$$du = e^t dt \qquad\qquad v = -\cos t$$

$$I = uv - \int v\, du$$

$$I = -e^t \cos t + \int e^t \cos t\, dt$$

$$I_1 = \int e^t \cos t\, dt$$

$$u = e^t \qquad\qquad dv = \cos t\, dt$$

$$du = e^t\, dt \qquad\qquad v = \sin t$$

$$I_1 = e^t \sin t - \int e^t \sin t\, dt = e^t \sin t - I$$

$$I = -e^t \cos t + e^t \sin t - I$$

$$2I = e^t \sin t - e^t \cos t + c_1$$

$$I = \frac{1}{2} (e^t \sin t - e^t \cos t) + c$$

$$S = 2\sqrt{5}\,\pi \left[\frac{1}{2} (e^t \sin t - e^t \cos t)\right]_0^{\pi} = \sqrt{5}\,\pi\, [0 + e^{\pi} - (0 - e^0)] = \sqrt{5}\,\pi\, [1 + e^{\pi}]$$

Example 5

24 May 2005 A

Let C be the curve given by parametric equations:

$$x(t) = \cosh\sqrt{t} \quad , \qquad y(t) = 1 + \sqrt{t} \quad , \qquad 1 \le t \le 9.$$

(a) Find the length of C.
(b) Find the slope of the tangent line to C at the point corresponding to $t = 4$

Solution

$$x = \cosh\sqrt{t}$$
$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}\sinh\sqrt{t}$$

$$y = 1 + \sqrt{t}$$
$$\frac{dy}{dt} = \frac{1}{2\sqrt{t}}$$

(a)

$$\left(\frac{dx}{dt}\right)^2 = \frac{1}{4t}\sinh^2\sqrt{t}$$
$$\left(\frac{dy}{dt}\right)^2 = \frac{1}{4t}$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\; dx$$

$$L = \int_{1}^{9} \sqrt{\frac{1}{4t} + \frac{1}{4t}\sinh^2\sqrt{t}}\; dt = \int_{1}^{9} \frac{1}{2\sqrt{t}}\sqrt{\left(1 + \sinh^2\sqrt{t}\right)}\; dt = \int_{1}^{9} \frac{1}{2\sqrt{t}}\cosh\sqrt{t}\; dt$$

$$= \left[\sinh\sqrt{t}\right]_1^9 = \sinh 3 - \sinh 1$$

$b)$

$\underline{m}$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{1}{2\sqrt{t}} \cdot \frac{2\sqrt{t}}{\sinh\sqrt{t}} = \operatorname{csch}\sqrt{t}$$

$$m = \left.\frac{dy}{dx}\right|_{t=4} = \operatorname{csch} 2$$

$\underline{p}$

$$x|_{t=4} = \cosh 2$$
$$y|_{t=4} = 1 + \sqrt{4} = 3$$

$$m = \operatorname{csch} 2 \qquad , \qquad p(\cosh 2\ , 3\ )$$
$$y - y_1 = m\,(x - x_1)$$
$$y - 3 = \operatorname{csch} 2\,(x - \cosh 2)$$

Homework

1	Find the length of the curve : $x = t - 2\tanh\frac{t}{2}$, $y = \operatorname{sech}\frac{t}{2}$, $0 \le t \le 4$	
2	Find the arc length of the curve $x = \cos t$, $y = t + \sin t$ $0 \le t \le \pi$	
3	Find the arc length of the curve defined parametrically by the equations $x = -\cos t + 5$, $y = \sin t + t - \frac{\pi}{2}$, $0 \le t \le \frac{\pi}{2}$	1 January 1995
4	Find the arc length of the curve $x = \sin t$, $y = \frac{1}{4}\cos 2t$ $0 \le t \le \frac{\pi}{2}$	2 February 1995
5	Find the arc length of the curve defined parametrically by the equations $x = t - \tanh t$, $y = \operatorname{sech} t$, $0 \le t \le 2$	10 August 1997
6	Find the equation of the tangent line to the curve given parametrically by $x(t) = t^2 + t + 1$, $y(t) = \frac{t^3}{3} - \frac{t^2}{2} + 2$ at the point $t = 1$. Also find the points where the tangent line is parallel to the $x - axis$	11 September 1997
7	Find the length of the curve given parametrically by $x(t) = e^t$, $y(t) = t$, $0 \le t \le \ln\sqrt{3}$	13 August 1998
8	Determine whether the statement is true or false. Justify your answer (1.5 pts. Each) The curve given by the parametric equation $x = 2\sin^2 t$, $y = 3\cos^2 t$ for $0 \le t \le \frac{\pi}{2}$ is a line segment from (0 , 3) to (2 , 0)	35 January 24, 2010
9	(4 pts.) Consider the parametric equations $x = t\cos t$, $y = t\sin t$ for $t \in [0, 2\pi]$. Find an equation of the tangent line at $t = 3\pi/2$.	36 June 6, 2010
10	Let C be the curve given by $x = \frac{2}{3}(1-t)^{3/2}$, $y = 2(1+t)^{1/2}$, $0 \le t \le 1$. (a) Find d^2y/dx^2.is C concave upward or downward ? (3 points) (b) Find the area of the surface obtained by rotating C about the $x -$ axis (3 points)	37 August 7, 2010